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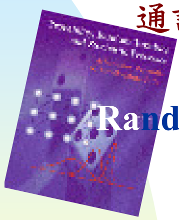
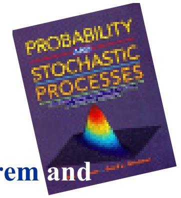
通訊與導航工程系碩士班

隨機過程



Random/Stochastic Processes
Fall 2004



吳家琪 助理教授

Lecture 10: Central Limit Theorem and the Sample Mean

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Announcement

- Course webpage:
  http://dcstl.cge.ntou.edu.tw/DCSTL/Web/rp.htm
- Reading Assignment:
  - Chapter 7 (Papoulis)
  - Chapters 7 and 8 (Yates)
- Homework No. 5 due today!

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Central Limit Theorem

- Review: $X_1, X_2, \ldots$ iid Gaussian rv's
- $W_n = X_1 + \ldots + X_n$ is Gaussian with

$$E[W_n] = n\mu_X$$
$$Var[W_n] = n\sigma_X^2$$

- What if $X_1, X_2, \ldots$ are not Gaussian?

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Sum of Bernoulli RV's

- 50 flips of a fair coin: $X_i = 1$ is H on flip i.
- W_n is binomial

$$P_{W_n}(w) = \begin{cases} \binom{50}{w}\left(\frac{1}{2}\right)^{50} & w = 0,1,\ldots,50 \\ 0 & otherwise \end{cases}$$

- What does this look like?

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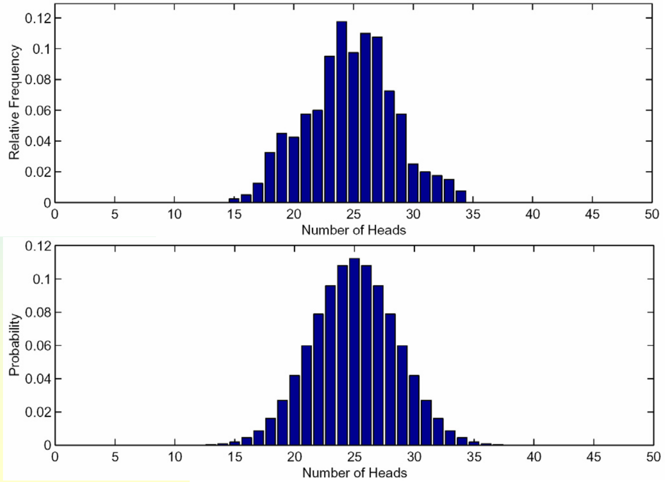
Sum of Bernoulli RV's



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Central Limit Theorem

- Standardized rv Z_n:

$$Z_n = \frac{W_n - E[W]}{\sigma_W} = \frac{\sum_{i=1}^{n} X_i - n\mu_X}{\sqrt{n\sigma_X^2}}$$

- $E[Z_n] = 0$, $Var[Z_n] = 1$
- Central Limit Theorem:

$$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z)$$

- Usual Proof: Show MGF of Zn converges to Gaussian MGF

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Applying the CLT

- For $W_n = X_1 + \ldots + X_n$,

$$F_{W_n}(w) = P\left[\sqrt{n\sigma_X^2}\, Z_n + n\mu_X \le w\right] = F_{Z_n}\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right)$$

- For large n, CLT says $F_{Z_n}(z) \approx \Phi(z)$.
- CLT Approximation:

$$F_{W_n}(w) \approx \Phi\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right)$$

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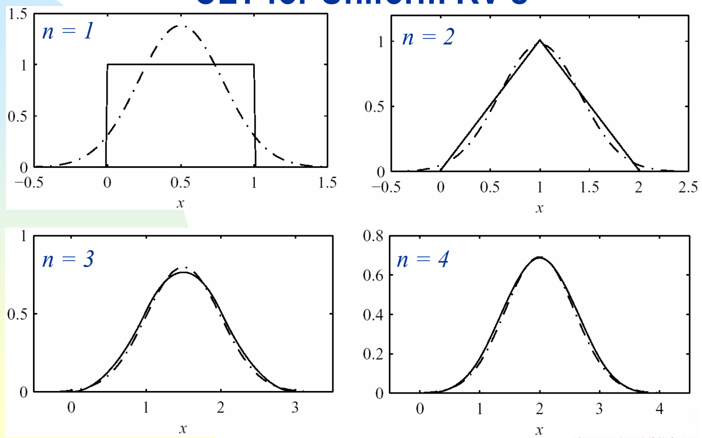
CLT for Uniform RV's



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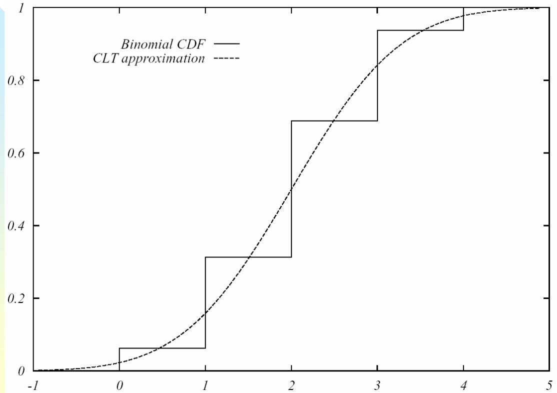
CLT for Binomial RV's



$n = 4,\ \ p = 1/2$

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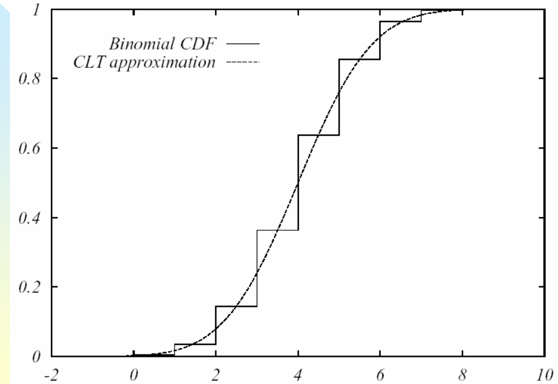
CLT for Binomial RV's



$n = 8,\ \ p = 1/2$

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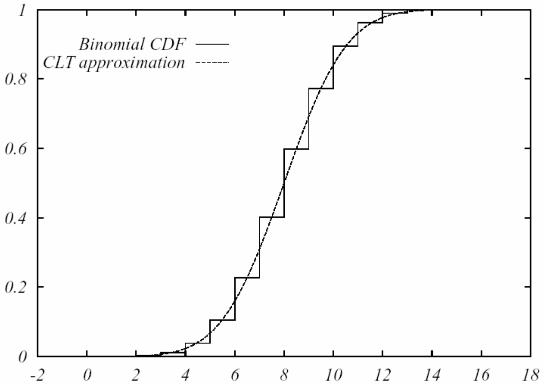
CLT for Binomial RV's



$n = 16,\ \ p = 1/2$

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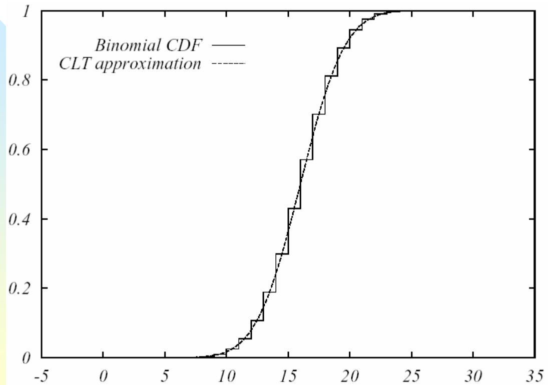
CLT for Binomial RV's



$n = 32,\ \ p = 1/2$

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Chap. 8: Sample Mean

- iid rv's $X_1, \ldots, X_n$ each with PDF $f_X(x)$
- The *sample mean* of X is the RV

$$M_n(X) = \frac{X_1 + \ldots + X_n}{n}$$

- Remember $M_n(X)$ is a RV!
- $M_n(X)$ is **not** the expected value $E[X]$

Mean and Variance of $M_n(X)$

- Theorem:

$$E[M_n(X)] = E[X] \qquad Var[M_n(X)] = \frac{Var[X]}{n}$$

- $\lim_{n \to \infty} Var[M_n(X)] = 0$ suggests $M_n(X) \to E[X]$
- How does a sequence of RV's approach a constant?

Markov Inequality

- For nonnegative rv X and $c > 0$,

$$P[X \geq c] \leq \frac{E[X]}{c}$$

- Proof: $f_X(x) = 0$ for $x < 0$ and

$$E[X] = \int_0^c x f_X(x)dx + \int_c^\infty x f_X(x)dx$$

$$\geq \int_c^\infty x f_X(x)dx \geq c\int_c^\infty f_X(x)dx = cP[X \geq c]$$

Markov Inequality Example

- X = height (in feet) of a random adult.
- $E[X] = 5.5$ ft
- Markov inequality says

$$P[X \geq 11] \leq 5.5/11 = 1/2$$

- Statement is true but is so weak it sounds wrong

Chebyshev Inequality

- Let $X = (Y - \mu_Y)^2$ and apply the Markov inequality:

$$P[(Y-\mu_Y)^2 \geq c^2] \leq \frac{E[(Y-\mu_Y)^2]}{c^2}$$

- Chebyshev Inequality:

$$P[|Y-\mu_Y| \geq c] \leq \frac{Var[Y]}{c^2}$$

Chernoff Bound

- For rv X and constant c,

$$P[X \geq c] \leq \min_{s \geq 0} e^{-sc}\phi_X(s)$$

- Proof:

$$P[X \geq c] = \int_c^\infty f_X(x)dx = \int_{-\infty}^\infty u(x-c) f_X(x)dx$$

- For all $s \geq 0$, $u(x-c) \leq e^{s(x-c)}$, implying

$$P[X \geq c] \leq \int_{-\infty}^\infty e^{s(x-c)} f_X(x)dx$$

$$= e^{-sc}\int_{-\infty}^\infty e^{sx} f_X(x)dx = e^{-sc}\phi_X(s)$$

- Upper bound must hold for minimizing s

Heights: Chebyshev

- For height X, $E[X] = 5.5$ and $\sigma_X = 1$ ft

$P[X \geq 11] = P[X-\mu_X \geq 11-\mu_X] = P[|X-\mu_X| \geq 5.5]$

- Chebyshev:

$P[X \geq 11] = P[|X-\mu_X| \geq 5.5]$

$\leq Var[X]/(5.5)^2 = 0.033 \approx 1/30$

Heights: Chernoff

- If X is $N[5.5, 1]$, $\phi_X(s) = e^{(11s+s^2)/2}$
- The Chernoff bound is

$$P[X \geq 11] \leq \min_{s \geq 0} e^{-11s}e^{(11s+s^2)/2} = \min_{s \geq 0} e^{(s^2-11s)/2}$$

- Choose s to min $h(s) = s^2 - 11s$ ➔ $s = 5.5$ and

$$P[X \geq 11] \leq e^{(s^2-11s)/2}\Big|_{s=5.5} = e^{-(5.5)^2/2} = 2.7 \times 10^{-7}$$

Inequalities

- Markov: For nonnegative X and $c > 0$,

$$P[X \geq c] \leq \frac{E[X]}{c}$$

- Chebyshev:

$$P[|Y - \mu_Y| \geq c] \leq \frac{Var[Y]}{c^2}$$

- Chernoff:

$$P[X \geq c] \leq \min_{s \geq 0} e^{-sc} \phi_X(s)$$

Chebyshev for $M_n(X)$

- For any $c > 0$,
  - $P[|M_n(X) - \mu_X| \geq c] \leq \frac{Var[X]}{nc^2} = \alpha$
  - $P[|M_n(X) - \mu_X| < c] \geq 1 - \frac{Var[X]}{nc^2} = 1 - \alpha$

Chebyshev for $M_n(X)$

- $P[|M_n(X) - \mu_X| < c] \geq 1 - \frac{Var[X]}{nc^2} = 1 - \alpha$
- Prob. the sample mean is within c of the expected value less than $\frac{Var[X]}{nc^2}$
- c = size of confidence interval
- $\alpha = Var[X]/nc^2$ is the confidence coefficient
- Small α : high confidence

Voter Survey

- Out of 1103 voters, the percentage supporting Jones is 58% ±3 percentage points.
- In this case, the data provides an estimate $M_n(X)$=0.58. What is the confidence coefficient α of this statement?

Voter Survey

- Experiment: observe whether a random voter supports Jones.
- $X = 1$ if she supports Jones, and $X = 0$ otherwise
- X is a Bernoulli: $E[X] = p$, $Var[X] = p(1–p)$
- For $c = 0.03$, **Theorem 8.5(b)** says

$$P[|M_n(X) - p| < 0.03] \geq 1 - \frac{p(1-p)}{n(0.03)^2} = 1 - \alpha$$

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Voter Survey

- confidence is $\alpha = \dfrac{p(1-p)}{n(0.03)^2}$
- For all p, $Var[X] = p(1–p) \leq 0.25$,

$$\alpha \leq \frac{0.25}{n(0.03)^2} = \frac{277.778}{n} \Rightarrow n \geq 1103$$

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Averaged Measurements

- X_i is ith independent measurement: $X_i = b+Z_i$
- Z_i is random, $E[Z_i] = 0$, $\sigma_Z = 1$
- Use $M_n(X)$ to get accurate estimate
- What n guarantees with a probability of $1-\alpha = 0.99$, or higher, that the estimate is within 0.1cm of the exact length of the board?

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Averaged Measurements

- $E[X_i] = b$, $Var[X_i] = Var[Z] = 1$,

$$P[|M_n(X) - b| < 0.1] \geq 1 - \frac{1}{n(0.1)^2} = 1 - \frac{100}{n}$$

- $P[|M_n(X) - b| < 0.1] \geq 0.99$ if $100/n \leq 0.01$.
- we need $n \geq 10{,}000$ measurements.
- By knowledge of mean value and variance of the prob. model of measurement errors, 10000 times are required to have an accuracy measurement.

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Averaged Measurements

- Find n if Z_i are iid Gaussian
- $M_n(X) = b+1/n(Z_1+ \ldots + Z_n) = b+M_n(Z)$ is Gaussian with

  $E[M_n(Z)] = b, \quad Var[M_n(Z)] = Var[Z]/n = 1/n$
- Thus

  $$P[|M_n(X)-b|<0.1] = 1-[\Phi(0.1\sqrt{n}) - \Phi(-0.1\sqrt{n})] = 2-2\Phi(\sqrt{n}/10).$$
- For $P[|M_n(X)-b|<0.1] \le 0.01$, we need

  $$2-2\Phi(\sqrt{n}/10) \le 0.01,$$

Averaged Measurements

- That is, $\Phi(\sqrt{n}/10) \ge 0.995$.
- Table 4.1 indicates $\Phi(x) \ge 0.995 \rightarrow x \ge 2.58$
- Therefore, $\sqrt{n}/10 \ge 2.58$, or $n \ge 666$.
- With knowledge of entire prob. model, we learn that only 666 measurements are necessary to guarantee an accuracy condition.

Weak Law of Large Numbers

- For any $c > 0$,
  - $\lim_{n\to\infty} P[|M_n(X) - \mu_X| \ge c] = 0$
  - $\lim_{n\to\infty} P[|M_n(X) - \mu_X| < c] = 1$

Relative Frequencies

- n independent trials, event A indicator:

  $$X_i = \begin{cases} 1 & \text{if } A \text{ occurs on trial } i \\ 0 & \text{otherwise} \end{cases}$$
- $E[X_i] = P[A]$, $Var[X_i] = P[A](1-P[A])$
- Relative frequency of A is

  $$R_n = M_n(X) = \frac{X_1 + \ldots + X_n}{n}$$
- Weak law says $\lim_{n\to\infty} P[|R_n - P[A]| \ge c] = 0$

Chebyshev for $M_n(X)$

- For any $c > 0$,
  - $P[|M_n(X) - \mu_X| \geq c] \leq \dfrac{Var[X]}{nc^2}$
  - $P[|M_n(X) - \mu_X| < c] \geq 1 - \dfrac{Var[X]}{nc^2}$
- **Weak Law** For any $c > 0$,
  - $\lim_{n \to \infty} P[|M_n(X) - \mu_X| \geq c] = 0$
  - $\lim_{n \to \infty} P[|M_n(X) - \mu_X| < c] = 1$

Convergence

- **Deterministic Sequence:** $a_1, a_2, \ldots \to a$ if given any $\delta > 0$, there exists n_0 such that for all $n \geq n_0$,

$$|a_n - a| \leq \delta.$$

- Y_n converges in probability to y if for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P[|Y_n - y| \geq \varepsilon] = 0.$$

- If $a_n = P[|Y_n - y| \geq \varepsilon]$, then $Y_n \to y$ in probability iff $a_n \to 0$ deterministically.

Sample Function Convergence

- Outcome s maps to sample sequence $y_1(s), y_2(s), y_3(s), \ldots$
- Probability a sample function of $Y_1, Y_2, \ldots$ converges to y is

$$P[C_y] = P[\{s \in S \mid \lim_{n \to \infty} y_n(s) = y\}]$$

- Note: If $\lim_{n \to \infty} y_n(s)$ doesn't exist, then $s \notin C_y$

Convergence with Probability 1

- $Y_1, Y_2, \ldots$ *converges to y with probability 1*, if and only if $P[C_y] = 1$. For this type of process we use the notation

$$\lim_{n \to \infty} Y_n = y \quad \text{w.p. } 1$$

- Theorem: $Y_n \to y$ w.p. 1 iff for all $\varepsilon > 0$,

$$\lim_{n \to \infty} P[|Y_n - y| < \varepsilon], |Y_{n+1} - y| < \varepsilon, \ldots] = 1$$

Testing Convergence w.p. 1

- **Definition:** The prob. that eventually Y_n, Y_{n+1}, … are all close to y equals 1.
- **Theorem:** Eventually, the prob. that Y_n, Y_{n+1}, … are all close to y converges to 1.

Example: w.p. 1

- U_1, U_2, … are iid uniform [0,1]
- Show $Y_n = \max\{U_1, \ldots, U_n\} \rightarrow 1$ w.p.1

Proof:

- Since $Y_n = \max\{U_n, Y_{n-1}\}$ so $Y_n \geq Y_{n-1}$ and

$$P[Y_n \geq 1-\varepsilon, Y_{n+1} \geq 1-\varepsilon, \ldots] = P[Y_n \geq 1-\varepsilon] = 1-(1-\varepsilon)^n$$

and $\lim_{n\to\infty} P[Y_n \geq 1-\varepsilon, Y_{n+1} \geq 1-\varepsilon, \ldots] = 1$

Lottery Example: Conv in Prob

- In a 50/50 lottery, each ticket costs \$1.
- 1 of n tickets picked randomly, winner gets \$$n/2$

- W_n has PMF

$$P_{W_n}(w) = \begin{cases} (n-1)/n & w = 0 \\ 1/n & w = n/2 \\ 0 & otherwise \end{cases}$$

Lottery Example: Conv in Prob

- For $0 < \varepsilon < n/2$,
- $P[|W_n - 0| < \varepsilon] = P[W_n = 0] = (n-1)/n$
- Hence $W_n \rightarrow 0$ in probability since

$$\lim_{n\to\infty} P[|W_n - 0| < \varepsilon] = \lim_{n\to\infty} \frac{n-1}{n} = 1$$

Lottery Example: w.p.1 ?

- For $0 < \varepsilon < 1/2$,

$$P[|W_n - 0| < \varepsilon, |W_{n+1} - 0| < \varepsilon, \ldots] = \lim_{j\to\infty} P[W_n = 0, W_{n+1} = 0, \ldots, W_j = 0]$$

- Implying

$$\lim_{j\to\infty} P[W_n = 0, W_{n+1} = 0, \ldots, W_j = 0] = \lim_{j\to\infty} P_{W_n}(0)P_{W_{n+1}}(0)\ldots P_{W_j}(0) = \lim_{j\to\infty} \frac{n-1}{n}\,\frac{n}{n+1}\cdots\frac{j-2}{j-1}\,\frac{j-1}{j} = \lim_{j\to\infty}\frac{n-1}{j} = 0$$

- Thus W_n does *not* converge to zero w.p.1

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Strong Law of Large Numbers

- If $X_1, X_2, \ldots$ is an independent identically distributed random sequence with sample mean $M_n(X)$,

$$\lim_{n\to\infty} M_n(X) = E[X] \text{ w.p. } 1$$

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